Entanglement Zoo I: Foundational and Structural Aspects

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Abstract

We develop a general framework for a structural description of the entanglement present in composite entities experimentally violating Bell's inequalities. Our formalism enables quantum modeling in complex Hilbert space for different types of situations, namely, situations where entangled states and product measurements appear ('standard quantum modeling'), and situations where both states and measurements are entangled ('nonlocal box modeling', 'nonlocal non-marginal box modeling'). The role played by Tsirelson's bound and marginal distribution law is emphasized. Specific quantum models are worked out in detail in complex Hilbert space within the present framework.

Keywords: Quantum modeling, Bell's inequalities, entanglement, nonlocal boxes

1 Introduction

Entanglement is one of the most fascinating, intriguing, mysterious and controversial aspects of quantum physics. It is the feature that most neatly marked the departure from ordinary intuition and common sense, on which classical physics rest. The structural and conceptual novelties brought in by quantum entanglement were originally put forward by John Bell in 1964. He proved that, if one introduces 'reasonable assumptions for physical theories', one derives an inequality for the expectation values of coincidence measurements performed on composite entities ('Bell's inequality') which does not hold in quantum theory [1]. Entanglement, being responsible for the violation of this inequality, entails that quantum particles share statistical correlations that cannot be described in a single classical Kolmogorovian probability framework [2, 3, 4]. Another highly amazing outcome was that entanglement, together with a number of other quantum features, such as 'contextuality', 'emergence', 'interference' and 'superposition', also appears outside the microscopic domain of quantum theory, in the dynamics of concepts and decision processes within human thought, in computer science, in biological interactions, etc. These findings constituted the beginning of a systematic and promising search for quantum structures and the employment of quantum-based models in domains where classical structures prove to be problematical [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

As for our own research, many years ago we already identified situations in macroscopic physics which violate Bell's inequalities [17, 18, 19, 20, 21]. One of these macroscopic examples, the 'connected vessels of water', exhibits even a maximal possible violation of Bell's inequalities, i.e. more than the typical entangled spin example in quantum physics. More recently, we performed a cognitive experiment showing that a specific combination of concepts, *The Animal Acts*, violates Bell's inequalities [22, 23, 24]. These two situations present deep structural and conceptual analogies which we analyze systematically in Ref. [25, 26].

In the present paper, we develop a unified framework that enables us to represent experimental situations of composite entities which violate Bell's inequalities identifying the quantum-theoretic modeling involved in these violations. We show that a complete quantum-mechanical representation (states, measurements, Born rule, state transformations, tensor product) can be worked out, and we prove that quantum entanglement is a 'joint feature' of states 'and' measurements. Indeed, we show that the empirical data we collected (Sec. 2), as well as the situation of the 'connected vessels of water' [26], can be modeled only when both states and measurements are entangled. Moreover, the marginal distribution law imposes serious restrictions on how this entanglement can be distributed on states and measurements. In this way, we recover within our general approach the quantum-theoretic modeling of different types of situations and entities represented in a complex Hilbert space.

(i) Situations where Bell's inequalities are violated within 'Tsirelson's bound' [27] and the marginal distribution law holds ('standard quantum modeling'). In this case, entangled states and product measurements occur (Sec. 3).

(ii) Situations where Bell's inequalities are violated within Tsirelson's bound and the marginal distribution law is violated ('nonlocal non-marginal box modeling 1'). In this case, both entangled states and entangled measurements occur (Sec. 3).

(iii) Situations where Bell's inequalities are violated beyond Tsirelson's bound and the marginal distribution law is violated ('nonlocal non-marginal box modeling 2'). In this case, both entangled states and entangled measurements occur (Sec. 3).

(iv) Situations where Bell's inequalities are violated beyond Tsirelson's bound and the marginal distribution law holds ('nonlocal box modeling'). In this case, both entangled states and entangled measurements occur (Sec. 4).

We recall that situations of type (ii) seem to be present in 'real quantum spin experiments' (a reference to the 'experimental anomaly' that, in our opinion, indicates the presence of entangled measurements, occurs already in Alain Aspects PhD thesis [28, 29]). Our framework accommodates these situations too.

Additionally to introducing the framework, we analyze in this paper the hypothesis that 'satisfying the marginal distribution law' is merely a consequence of extra symmetry being present in situations that contain full-type entanglement, e.g., situations of types (ii) and (iii). Whenever enough symmetry is present, such that all the entanglement of the situation can be pushed into the state, allowing a model with product measurements, the marginal law is satisfied. We give two examples, a cognitive 'gedanken experiment' violating Bell's inequalities, which is a 'variation adding more symmetry' to an example that was introduced in Ref. [21], and in this variation the marginal law is satisfied. We introduce in a similar way extra symmetry in our 'vessels of water example, to come to a variation where the marginal law is satisfied. Both examples are isomorphic and realizations of the so-called 'nonlocal box', which is studied as a purely theoretical construct – no physical realizations were found prior to the ones we present here – in the foundations of quantum theory [30] (Sec. 4).

2 Technical aspects of the quantum modeling of entanglement

To develop a general approach to the study of entanglement for bipartite entities, we will first introduce some basic notions and results.

Let us consider the canonical orthonormal (ON) bases $\{|1,0\rangle, |0,1\rangle\}$ and $\{|1,0,0,0\rangle, |0,1,0,0\rangle, |0,0,1,0\rangle, |0,0,0,1\rangle\}$ in $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively. A canonical isomorphism between $\mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$ is defined by

$$|1,0,0,0\rangle \leftrightarrow |1,0\rangle \otimes |1,0\rangle \quad |0,1,0,0\rangle \leftrightarrow |1,0\rangle \otimes |0,1\rangle \tag{1}$$

$$|0,0,1,0\rangle \leftrightarrow |0,1\rangle \otimes |1,0\rangle \quad |0,0,0,1\rangle \leftrightarrow |0,1\rangle \otimes |0,1\rangle \tag{2}$$

Let us also recall that the vector space $\mathcal{L}(\mathbb{C}^4)$ of all linear operators on $\mathbb{C}^4$ is isomorphic to the vector space $\mathcal{L}(\mathbb{C}^2)\otimes\mathcal{L}(\mathbb{C}^2)$, where $\mathcal{L}(\mathbb{C}^2)$ is the set of all linear operators on $\mathbb{C}^2$. The canonical isomorphism defined by (1) and (2) introduces hence a corresponding canonical isomorphism between $\mathcal{L}(\mathbb{C}^4)$ and $\mathcal{L}(\mathbb{C}^2)\otimes\mathcal{L}(\mathbb{C}^2)$. Both canonical isomorphisms will be denoted by $\leftrightarrow$ from now on. Let us introduce the notions of 'product state' and 'product measurement'.

Definition 1. *A state p, represented by the unit vector $|p\rangle \in \mathbb{C}^4$, is a 'product state' if there exist two states p_1 and p_2, represented by the unit vectors $|p_1\rangle \in \mathbb{C}^2$ and $|p_2\rangle \in \mathbb{C}^2$, respectively, such that $|p\rangle \leftrightarrow |p_1\rangle \otimes |p_2\rangle$. Otherwise, p is an 'entangled state'.*

Definition 2. *A measurement e, represented by a self-adjoint operator $\mathcal{E}$ in $\mathbb{C}^4$, is a 'product measurement' if there exist measurements e_1 and e_2, represented by the self-adjoint operators $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, in $\mathbb{C}^2$, such that $\mathcal{E} \leftrightarrow \mathcal{E}_1 \otimes \mathcal{E}_2$. Otherwise, e is an 'entangled measurement'.*

Let now p be a product state, represented by $|p_1\rangle \otimes |p_2\rangle$, where $|p_1\rangle$ and $|p_2\rangle$ represent the states p_1 and p_2, respectively. And let e be a product measurement, represented by $\mathcal{E}_1 \otimes \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ represent the measurements e_1 and e_2, respectively. The following theorems can then be proved.

Theorem 1. *The spectral family of the self-adjoint operator $\mathcal{E}_1 \otimes \mathcal{E}_2$ representing the product measurement e has the form $|p_{11}\rangle\langle p_{11}| \otimes |p_{21}\rangle\langle p_{21}|$, $|p_{11}\rangle\langle p_{11}| \otimes |p_{22}\rangle\langle p_{22}|$, $|p_{12}\rangle\langle p_{12}| \otimes |p_{21}\rangle\langle p_{21}|$ and $|p_{12}\rangle\langle p_{12}| \otimes |p_{22}\rangle\langle p_{22}|$, where $|p_{11}\rangle\langle p_{11}|$ and $|p_{12}\rangle\langle p_{12}|$ is a spectral family of $\mathcal{E}_1$ and $|p_{21}\rangle\langle p_{21}|$ and $|p_{22}\rangle\langle p_{22}|$ is a spectral family of $\mathcal{E}_2$.*

Theorem 1 states that the spectral family of a product measurement is made up of product orthogonal projection operators.

Theorem 2. *Let p be a product state represented by $|p_1\rangle \otimes |p_2\rangle$, and e a product measurement represented by $\mathcal{E}_1 \otimes \mathcal{E}_2$. Then, there exist probabilities $p(\lambda_{A_1})$, $p(\lambda_{B_1})$, $p(\lambda_{A_2})$ and $p(\lambda_{B_2})$, where $p(\lambda_{A_i})$ ($p(\lambda_{B_i})$) is the probability for the outcome λ_{A_i} (λ_{B_i}) of e_1 (e_2) in the state p_1 (p_2), $i = 1, 2$, such that $p(\lambda_{A_1}) + p(\lambda_{A_2}) = p(\lambda_{B_1}) + p(\lambda_{B_2}) = 1$, and $p(\lambda_{A_iB_j}) = p(\lambda_{A_i})p(\lambda_{B_j})$, where $\lambda_{A_iB_j}$, $i, j = 1, 2$ are the outcomes of e in the state p.*

An immediate consequence of Th. 2 is that, whenever the probabilities $p(\lambda_{A_iB_j})$ do not factorize, only three possibilities exist: (i) the state p is not a product state; (ii) the measurement e is not a product measurement; (iii) both p and e are entangled.

Theorem 3. *A state p represented by the unit vector $|p\rangle = |ae^{i\alpha}, be^{i\beta}, ce^{i\gamma}, de^{i\delta}\rangle$ in the canonical basis of $\mathbb{C}^4$ is a product state iff $a \cdot d \cdot e^{i(\alpha+\delta)} - b \cdot c \cdot e^{i(\beta+\gamma)} = 0$.*

Let us consider the coincidence measurements e_{AB}, $e_{AB'}$, $e_{A'B}$ and $e_{A'B'}$ introduced in the experiments testing Bell's inequalities (Sec. 3). The measurement e_{AB} is associated with the outcomes $\lambda_{A_1B_1}$, $\lambda_{A_1B_2}$, $\lambda_{A_2B_1}$ and $\lambda_{A_2B_2}$, the measurement $e_{AB'}$ is associated with the outcomes $\lambda_{A_1B'_1}$, $\lambda_{A_1B'_2}$, $\lambda_{A_2B'_1}$ and $\lambda_{A_2B'_2}$, the measurement $e_{A'B}$ is associated with the outcomes $\lambda_{A'_1B_1}$, $\lambda_{A'_1B_2}$, $\lambda_{A'_2B_1}$ and $\lambda_{A'_2B_2}$, measurement $e_{A'B'}$ is associated with the outcomes $\lambda_{A'_1B'_1}$, $\lambda_{A'_1B'_2}$, $\lambda_{A'_2B'_1}$ and $\lambda_{A'_2B'_2}$. Each outcome $\lambda_{A_1B_1}$, $\ldots$, $\lambda_{A'_2B'_2}$ is associated with a defined probability $p(\lambda_{A_1B_1})$, $\ldots$,$p(\lambda_{A'_2B'_2})$ in a given state p.

Definition 3. *We say that a set of experimental data $\mathcal{S}_{AB}$ collected during the measurement e_{AB} satisfies the 'marginal distribution law' if*

$$p(\lambda_{A_1B_1}) + p(\lambda_{A_1B_2}) = p(\lambda_{A_1B'_1}) + p(\lambda_{A_1B'_2}) \tag{3}$$
$$p(\lambda_{A_2B_1}) + p(\lambda_{A_2B_2}) = p(\lambda_{A_2B'_1}) + p(\lambda_{A_2B'_2}) \tag{4}$$
$$p(\lambda_{A_1B_1}) + p(\lambda_{A_2B_1}) = p(\lambda_{A'_1B_1}) + p(\lambda_{A'_2B_1}) \tag{5}$$
$$p(\lambda_{A_1B_2}) + p(\lambda_{A_2B_2}) = p(\lambda_{A'_1B'_2}) + p(\lambda_{A'_2B_2}) \tag{6}$$

We say that the marginal probability distribution is satisfied in a Bell test if it is satisfied by all measurements e_{AB}, $e_{AB'}$, $e_{A'B}$ and $e_{A'B'}$.

Theorem 4. *Let e be a product measurement. Then, the marginal distribution law is satisfied by e.*

Theorem 5. *If no measurement among e_{AB}, $e_{AB'}$, $e_{A'B}$ and $e_{A'B'}$ satisfies the marginal distribution law, then at least two measurements are entangled.*

We refer to Ref. [25] for proof of Theorems 1–5. The latter provide a sharp and complete description of the structural situation, namely that 'entanglement is a relational property of states and measurements'. If Th. 2 is not satisfied by a set of experimental data collected during a single measurement, then one can transfer all the entanglement to the state, or to the measurement, or to both. Theorem 5 then shows that, whenever the marginal distribution law is violated, no more than two measurements can be products. The main consequence is that, whenever a set of experimental data violate both Bell's inequalities and the marginal distribution law, it is impossible to work out a quantum-mechanical representation in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ which satisfies the data and where only the initial state is entangled while all measurements are products. We will make this more explicit in the next sections.

3 Examples of systems entailing entanglement

Some time ago, one of us proved that it is possible to construct a macroscopic entity violating Bell's inequalities in exactly the same way as a pair of spin-1/2 quantum particles in the singlet spin state when faraway spin measurements are performed [20]. We only sketch this example here to make our zoo collection as complete as possible, and refer to [20, 21] for a detailed presentation.

This mechanical entity simulates the singlet spin state of a pair of spin-1/2 quantum particles by means of two point particles P_1 and $P2$ initially located in the centers C_1 and C_2 of two separate unit spheres B_1 and B_2, respectively. The centers C_1 and C_2 remain connected by a rigid but extendable rod, which introduces correlations. We denote this state of the overall entity by p_s. A measurement $e_A(a)$ is performed on P_1 which consists in installing a piece of elastic of 2 units of length between the diametrically opposite points $-a$ and $+a$ of B_1. At one point, the elastic breaks somewhere and P_1 is drawn toward either $+a$ (outcome $\lambda_{A_1} = +1$) or $-a$ (outcome $\lambda_{A_2} = -1$). Due to the connection, P_2 is drawn toward the opposite side of B_2 as compared to P_1. Now, an analogous measurement $e_B(b)$ is performed on P_2 which consists in installing a piece of elastic of 2 units of length between the two diametrically opposite points $-b$ and $+b$ of B_2. The particle P_2 falls onto the elastic following the orthogonal path and sticks there. Next the elastic breaks somewhere and drags P_2 toward either $+b$ (outcome $\lambda_{B_1} = +1$) or $-b$ (outcome $\lambda_{B_2} = -1$). To calculate the transition probabilities, we assume there is a uniform probability of breaking on the elastics. The single and coincidence probabilities coincide with the standard probabilities for spin-1/2 quantum particles in the singlet spin state when spin measurements are performed along directions a and b. In particular, the probabilities for the coincidence counts $\lambda_{A_1B_1} = \lambda_{A_2B_2} = +1$ and $\lambda_{A_1B_2} = \lambda_{A_2B_1} = -1$ of the joint measurement $e_{AB}(a, b)$ in the state p_s are given by

$$p(p_s, e_{AB}(a, b), \lambda_{A_1B_1}) = p(p_s, e_{AB}(a, b), \lambda_{A_2B_2}) = \frac{1}{2} \sin^2 \frac{\gamma}{2} \tag{7}$$

$$p(p_s, e_{AB}(a, b), \lambda_{A_1B_2}) = p(p_s, e_{AB}(a, b), \lambda_{A_2B_1}) = \frac{1}{2} \cos^2 \frac{\gamma}{2} \tag{8}$$

respectively, where γ is the angle between a and b, in exact accordance with the quantum-mechanical predictions. Furthermore, this model leads to the same violation of Bell's inequalities as standard quantum theory. Hence, the 'connected spheres model' is structurally isomorphic to a standard quantum entity. This

means that it can be represented in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ in such a way that its initial state is the singlet spin, i.e. a maximally entangled state, and the measurements are products. Furthermore, the marginal distribution law holds and Bell's inequalities are violated within the Tsirelson's bound $2\sqrt{2}$, hence the connected spheres model is an example of a 'customary identified standard quantum modeling' in our theoretical framework.

The presence of entanglement in concept combination has recently also been identified in a cognitive test [22, 23, 24] and subsequently improved by elaborating a quantum Hilbert space modeling of it [25, 26]. We analyze it in the light of our general perspective in Sec. 2.

We describe the sentence *The Animal Acts* as a combination of the concepts *Animal* and *Acts*. We then consider two couples of exemplars of *Animal*, namely *Horse*, *Bear* and *Tiger*, *Cat*, and two pairs of exemplars of *Acts*, namely *Growls*, *Whinnies* and *Snorts*, *Meows*. We then introduce the coincidence measurements e_{HBGW}, e_{TCGW}, e_{HBSM} and e_{TCSM} for the conceptual combination *The Animal Acts*. In all measurements, we ask subjects to answer the question whether the given statement 'is a good example of' the concept *The Animal Acts*. In the measurement e_{HBGW}, participants choose among the four possibilities (1) *The Horse Growls*, (2) *The Bear Whinnies* – and if one of these is chosen we put $\lambda_{HG} = \lambda_{BW} = +1$ – and (3) *The Horse Whinnies*, (4) *The Bear Growls* – and if one of these is chosen we put $\lambda_{HW} = \lambda_{BG} = -1$. In the measurement e_{TCGW}, they choose among (1) *The Tiger Growls*, (2) *The Cat Whinnies* – and in case one of these is chosen we put $\lambda_{TG} = \lambda_{CW} = +1$ – and (3) *The Tiger Whinnies*, (4) *The Cat Growls* – and in case one of these is chosen we put $\lambda_{TW} = \lambda_{CG} = -1$. In the measurement e_{HBGW}, they choose among (1) *The Horse Snorts*, (2) *The Bear Meows* – and in case one of these is chosen we put $\lambda_{HS} = \lambda_{BM} = +1$ – and (3) *The Horse Meows*, (4) *The Bear Snorts* – and in case one of these is chosen we put $\lambda_{HS} = \lambda_{BM} = -1$. In the measurement e_{TCSM} participants choose among (1) *The Tiger Snorts*, (2) *The Cat Meows* – and in case one of these is chosen we put $\lambda_{TS} = \lambda_{CM} = +1$ – and (3) *The Tiger Meows*, (4) *The Cat Snorts* – and in case one of these is chosen we put $\lambda_{TM} = \lambda_{CS} = -1$.

We now evaluate the expectation values $E(A', B')$, $E(A', B)$, $E(A, B')$ and $E(A, B)$ associated with the coincidence measurements e_{HBGW}, e_{TCGW}, e_{HBSM} and e_{TCSM}, respectively, and insert them into the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequality [31]

$$-2 \le E(A', B') + E(A', B) + E(A, B') - E(A, B) \le 2. \tag{9}$$

We asked 81 subjects to choose among the above alternatives in the measurements e_{HBGW}, e_{TCGW}, e_{HBSM} and e_{TCSM}. If we denote by $p(HG)$, $p(BW)$, $p(HW)$, $p(BG)$, the probability that *The Horse Growls*, *The Bear Whinnies*, *The Horse Whinnies*, *The Bear Growls*, respectively, is chosen in the measurement e_{HBGW}, and so on in the other experiments, the probabilities are $p(HG) = 0.049$, $p(HW) = 0.630$, $p(BG) = 0.259$, $p(BW) = 0.062$, $p(HS) = 0.593$, $p(HM) = 0.025$, $p(BS) = 0.296$, $p(BM) = 0.086$, $p(TG) = 0.778$, $p(TW) = 0.086$, $p(CG) = 0.086$, $p(CW) = 0.049$, $p(TS) = 0.148$, $p(TM) = 0.086$, $p(CS) = 0.099$, $p(CM) = 0.667$, and the expectation values are

$$\begin{aligned}
E(A,B) = &\quad p(HG) + p(BW) - p(BG) - p(HW) = -0.7778 \\
E(A,B') = &\quad p(HS) + p(BM) - p(BS) - p(HM) = 0.3580 \\
E(A',B) = &\quad p(TG) + p(CW) - p(CG) - p(TW) = 0.6543 \\
E(A',B') = &\quad p(TS) + p(CM) - p(CS) - p(TM) = 0.6296
\end{aligned}$$

Hence, Eq. (9) gives $E(A', B') + E(A', B) + E(A, B') - E(A, B) = 2.4197$. This violation significantly proves the presence of a genuine structural entanglement between the concept *Animal* and the concept *Acts* in the combination *The Animal Acts*.

First of all, we note that the probabilities corresponding to the coincidence measurements cannot be factorized, hence a result stronger than the one in Th. 2 holds in this case. Let us prove this for the

measurement e_{HBGW}. There do not exist real numbers $a_1, a_2, b_1, b_2 \in [0,1]$, $a_1 + a_2 = 1$, $b_1 + b_2 = 1$, such that $a_1 b_1 = 0.05$, $a_2 b_1 = 0.63$, $a_1 b_2 = 0.26$ and $a_2 b_2 = 0.06$. Indeed, if we suppose that such numbers do exist, then, from $a_2 b_1 = 0.63$ follows that $(1 - a_1) b_1 = 0.63$, and hence $a_1 b_1 = 1 - 0.63 = 0.37$. This is in contradiction with $a_1 b_1 = 0.05$.

Secondly, we can see that Th. 4 does not hold, whereas Th. 5 does. Indeed, if we compare Eqs. (3)–(6) in Def. 3 with collected data, we realize that the marginal distribution law is never satisfied. This entails that a quantum representation which fits our data and where only the state is entangled, while all measurements are products, does not exist. But a representation which entails entangled measurements can be elaborated [25, 26]. In our quantum modeling in Hilbert space, the state of *The Animal Acts* is represented by a non-maximally entangled state, while all coincidence measurements are entangled. Since the violation of the CHSH inequalitiy we found satisfies Tsirelson's bound, this quantum modeling for the concept combination *The Animal Acts* is an example of a 'nonlocal non-marginal box modeling 1'.

Next we consider the 'vessels of water example [17, 18, 19]. Hence, two vessels V_A and V_B are interconnected by a tube T, with the vessels and the tube together containing 20 liters of transparent water. The measurements e_A and e_B consist in siphons S_A and S_B pouring out water from vessels V_A and V_B, respectively, and collecting the water in reference vessels R_A and R_B, where the volume of collected water is measured. If more than 10 liters are collected for e_A or e_B, we put $\lambda_{A_1} = +1$ or $\lambda_{B_1} = +1$, respectively, and if fewer than 10 liters are collected for e_A or e_B, we put $\lambda_{A_2} = -1$ or $\lambda_{B_2} = -1$, respectively. We define the measurements $e_{A'}$ and $e_{B'}$, which consist in taking a small spoonful of water out of the left vessel and the right vessel, respectively, and verifying whether the water is transparent. We have $\lambda_{A'_1} = +1$ or $\lambda_{A'_2} = -1$, depending on whether the water in the left vessel turns out to be transparent or not, and $E\lambda_{B'_1} = +1$ or $\lambda_{B'_2} = -1$, depending on whether the water in the right vessel turns out to be transparent or not. We put $\lambda_{A_1B_1} = \lambda_{A_2B_2} = +1$ if $\lambda_{A_1} = +1$ and $\lambda_{B_1} = +1$ or $\lambda_{A_2} = -1$ and $\lambda_{B_2} = -1$, and $\lambda_{A_1B_2} = \lambda_{A_2B_1} = -1$ if $\lambda_{A_1} = +1$ and $\lambda_{B_2} = -1$ or $\lambda_{A_2} = -1$ and $\lambda_{B_1} = +1$, if the coincidence measurement e_{AB} is performed. We proceed analogously for the outcomes of the measurements $e_{AB'}$, $e_{A'B}$ and $e_{A'B'}$. We can then define the expectation values $E(A,B)$, $E(A,B')$, $E(A',B)$ and $E(A',B')$ associated with the measurements e_{AB}, $e_{AB'}$, $e_{A'B}$ and $e_{A'B'}$, respectively. Since each vessel contains 10 liters of transparent water, we find that $E(A,B) = -1$, $E(A',B) = +1$, $E(A,B') = +1$ and $E(A',B') = +1$, which gives $E(A',B') + E(A',B) + E(A,B') - E(A,B) = +4$. This is the maximal violation of the CHSH inequality and it obviously exceeds Tsirelson's bound. We further have $0.5 = p(\lambda_{A_1B_1}) + p(\lambda_{A_1B_2}) \neq p(\lambda_{A_1B'_1}) + p(\lambda_{A_1B'_2}) = 1$. Thus, Th. 4 entails that the marginal distribution law is generally violated in the case of the vessels of water. We have constructed a quantum model in complex Hilbert space for the vessels of water situation [26]. In this model, both the state p with transparent water and the state q with non-transparent water are entangled. The measurement e_{AB} has product states in its spectral decomposition, hence it is a product measurement, because of Th. 1. The latter can be applied again to conclude that $e_{AB'}$, $e_{A'B}$ and $e_{AB'}$ are instead entangled measurements. Summarizing, we can say that the 'vessels of water situation is an example of a 'nonlocal non-marginal box modeling 2'.

4 Nonlocal boxes

We conclude this paper by giving two examples, the one physical and the other cognitive, which maximally violate Bell's inequalities, i.e. with value 4, but satisfy the marginal distribution law. In physics, a system that behaves in this way is called a 'nonlocal box' [30]. We will see that our examples are structurally isomorphic and exhibit the typical symmetry which gives rise to the marginal distribution law being valid in quantum theory.

Let us start by constructing a version of the vessels of water which maximally violates the CHSH inequality, but does not violate the marginal distribution law. This version is slightly different from the

original version of the vessels of water. Two measurements available for each side A and B of the vessels. The first consists in using the siphon and checking the water. If there are more than 10 liters and the water is transparent ($\lambda_{A_1B_1}$) or if there are fewer than 10 liters and the water is not transparent ($\lambda_{A_2B_2}$), the outcome of the first measurement will be $+1$. In case there are fewer than 10 liters and the water is transparent $\lambda_{A_2B_1}$, or if there are more than 10 liters and the water is not transparent $\lambda_{A_1B_2}$, the outcome will be -1. The second measurement consists in taking out some water with a little spoon to see if it is transparent or not; if it is transparent, the outcome is $\lambda_{A_1B'_1} = \lambda_{A_2B'_2} = +1$, and if it is not transparent, the outcome is $\lambda_{A_2B'_1} = \lambda_{A_1B'_2} = -1$. The water is prepared in a mixed state m of the states p (transparent water) and q (not transparent water) with equal weights. Thus, m is represented by the density operator $\rho = 0.5|p\rangle\langle p| + 0.5|q\rangle\langle q|$, where $|p\rangle = |0, \sqrt{0.5}e^{i\alpha}, 0.5e^{i\beta}, 0\rangle$ and $|q\rangle = |0, \sqrt{0.5}e^{i\alpha}, -0.5e^{i\beta}, 0\rangle$ [26].

The first measurement f_{AB} is represented by the ON set

$$|r_{A_1B_1}\rangle = |1,0,0,0\rangle \qquad |r_{A_1B_2}\rangle = |0,1,0,0\rangle \tag{10}$$

$$|r_{A_2B_1}\rangle = |0,0,1,0\rangle \qquad |r_{A_2B_2}\rangle = |0,0,0,1\rangle \tag{11}$$

which gives rise to a self-adjoint operator

$$\mathcal{F}_{AB} = \begin{pmatrix} \lambda_{A_1B_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_1B_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_2B_1} & 0 \\ 0 & 0 & 0 & \lambda_{A_2B_2} \end{pmatrix} \tag{12}$$

By applying Lüders' rule, we can now calculate the density operator representing the final state of the vessels of water after f_{AB}. This gives

$$\rho_{AB} = \sum_{i,j=1}^{2} |r_{A_iB_j}\rangle\langle r_{A_iB_j}|\rho|r_{A_iB_j}\rangle\langle r_{A_iB_j}| = \rho \tag{13}$$

as one can easily verify. This means that the nonselective measurement f_{AB} leaves the state m unchanged or, equivalently, the marginal distribution law holds.

The second measurement $f_{AB'}$ is instead represented by the ON set

$$|r_{A_1B'_1}\rangle = |0, \sqrt{0.5}e^{i\alpha}, \sqrt{0.5}e^{i\beta}, 0\rangle \qquad |r_{A_1B'_2}\rangle = |1,0,0,0\rangle \tag{14}$$

$$|r_{A_2B'_1}\rangle = |0,0,0,1\rangle \qquad |r_{A_2B'_2}\rangle = |0, \sqrt{0.5}e^{i\alpha}, -\sqrt{0.5}e^{i\beta}, 0\rangle \tag{15}$$

which gives rise to a self-adjoint operator

$$\mathcal{F}_{AB'} = \begin{pmatrix} \lambda_{A_1B'_2} & 0 & 0 & 0 \\ 0 & 0.5(\lambda_{A_1B'_1} + \lambda_{A_2B'_2}) & 0.5e^{i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_2B'_2}) & 0 \\ 0 & 0.5e^{-i(\alpha-\beta)}(\lambda_{A_1B'_1} - \lambda_{A_2B'_2}) & 0.5(\lambda_{A_1B'_1} + \lambda_{A_2B'_2}) & 0 \\ 0 & 0 & 0 & \lambda_{A_2B'_1} \end{pmatrix} \tag{16}$$

By applying Lüders' rule, we can again calculate the density operator representing the final state of the vessels of water after $f_{AB'}$. This gives

$$\rho_{AB'} = \sum_{i,j=1}^{2} |r_{A_iB'_j}\rangle\langle r_{A_iB'_j}|\rho|r_{A_iB'_j}\rangle\langle r_{A_iB'_j}| = \rho \tag{17}$$

Also in this case, the nonselective measurement $f_{AB'}$ leaves the state m unchanged. The measurements $f_{A'B}$ and $f_{A'B'}$ are analogous, hence we do not have to make the calculation each time the density operator

after applying Luders' rule remains the same. This implies that the marginal distribution law is always satisfied.

Let us now evaluate the expectation values corresponding to the four measurements above in the mixed state m and insert them into the CHSH inequality. The expectation value operators for this version are given by

$$\mathcal{F}_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{F}_{AB'} = \mathcal{F}_{A'B} = \mathcal{F}_{A'B'} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{18}$$

Hence, the Bell operator is given by

$$B = \mathcal{F}_{AB'} + \mathcal{F}_{A'B} + \mathcal{F}_{A'B'} - \mathcal{F}_{AB} = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \tag{19}$$

This gives in the CHSH inequality

$$tr\rho B = 4 \tag{20}$$

which shows that Bell inequalities are maximally violated in the mixed state m.

Let us now look at the cognitive example. We consider the concept *Cat* and two concrete exemplars of it, called *Glimmer* and *Inkling*, the names of two brother cats that lived in our research center [21]. The concept *Cat* is abstractly described by the state p. The experiments that we introduce consist in realizing physical contexts that influence the collapse of the concept *Cat* to one of its exemplars, or states, *Glimmer* or *Inkling*, inside the mind of a person being confronted with the physical contexts. It is a 'gedanken experiment', in the sense that we put forward plausible outcomes for it, taking into account the nature of the physical contexts, and Liane, the owner of both cats, playing the role of the person. We also suppose that Liane sometimes puts a collar with a little bell around the necks of both cats, and that sometimes she does not, but if she does, she always puts them around the necks of both, never around the neck of only one of them, the probability of either situation being equal to 1/2.

The measurement e_A consists in '*Glimmer* appearing in front of Liane as a physical context'. We consider outcome λ_{A_1} to occur if Liane thinks of *Glimmer* and there is a bell, or if she thinks of *Inkling* and there is no bell, while outcome λ_{A_2} occurs if Liane thinks of *Inkling* and there is a bell, or if she thinks of *Glimmer* and there is no bell. The measurement e_B consists in '*Inkling* appearing in front of Liane as a physical context'. We consider outcome λ_{B_1} to occur if Liane thinks of *Inkling* and there is a bell, or if she thinks of *Glimmer* and there is no bell, while outcome λ_{B_2} occurs if Liane thinks of *Glimmer* and there is a bell, or if she thinks of *Inkling* and there is no bell. Experiment $e_{A'}$ consists in '*Inkling* appearing in front of Liane as a physical context', and outcome $\lambda_{A'_1}$ occurs if Inkling wears a bell, and outcome $\lambda_{A'_2}$, if Inkling does not. Experiment $e_{B'}$ consists in '*Glimmer* appearing in front of Liane as a physical context', and outcome $\lambda_{B'_1}$ occurs if Glimmer wears a bell, outcome $\lambda_{B'_1}$, if Glimmer does not.

The measurement e_{AB} consists in both cats showing up as physical contexts. Because of the symmetry of the situation, it is plausible to suppose probability 1/2 that Liane thinks of *Glimmer* and probability 1/2 that she thinks of *Inkling*, however, they are mutually exclusive. Also, since both cats either wear bells or do not wear bells, e_{AB} produces strict anti-correlation, probability 1/2 for outcome $\lambda_{A_1B_2}$ and probability 1/2 for outcome $\lambda_{A_2B_1}$. Hence $p(\lambda_{A_1B_2}) = p(\lambda_{A_2B_1}) = 1/2$ and $p(\lambda_{A_1B_1}) = p(\lambda_{A_2B_2}) = 0$, which gives $E(A, B) = -1$. The measurement $e_{AB'}$ consists in *Glimmer* showing up as a physical context. This gives rise to a perfect correlation, outcome $\lambda_{A_1B'_1}$ or outcome $\lambda_{A_2B'_2}$, depending on whether *Glimmer* wears a bell or not, hence both with probability 1/2. As a consequence, we have $p(\lambda_{A_1B'_1}) = p(\lambda_{A_2B'_2}) = 1/2$ and

$p(\lambda_{A_1B'_2}) = p(\lambda_{A_2B'_1}) = 0$, and $E(A, B') = +1$. The measurement $e_{A'B}$ consists in *Inkling* showing up as a physical context, again giving rise to a perfect correlation, outcome $\lambda_{A'_1B_1}$ or outcome $\lambda_{A'_2B_2}$, depending on whether *Inkling* wears a bell or not, hence both with probability 1/2. This gives $p(\lambda_{A'_1B_1}) = p(\lambda_{A'_2B_2}) = 1/2$ and $p(\lambda_{A'_1B_2}) = p(\lambda_{A'_2B_1}) = 0$ and $E(A', B) = +1$. The measurement $e_{A'B'}$ consists in both cats showing up as physical contexts, giving rise to a perfect correlation, outcome $\lambda_{A'_1B'_1}$ or outcome $\lambda_{A'_2B'_2}$, depending on whether both wear bells or not, hence both with probability 1/2. This gives $p(\lambda_{A'_1B'_1}) = p(\lambda_{A'_2B'_2}) = 1/2$ and $p(\lambda_{A'_1B'_2}) = p(\lambda_{A'_2B'_1}) = 0$ and $E(A', B') = +1$.

We find $E(A', B') + E(A', B) + E(A, B') - E(A, B) = 4$ in the CHSH inequalitiy. The marginal distribution law is satisfied here, because, e.g., $p(\lambda_{A_1B_1}) + p(\lambda_{A_1B_2}) = p(\lambda_{A_1B'_1}) + p(\lambda_{A_1B'_2}) = 1/2$. It is easy to check that the marginal distribution law globally holds in this case.

The two examples above are structurally isomorphic, in the sense that one can provide the same quantum model in complex Hilbert space for both of them. Moreover, they are a realization of what quantum foundations physicists call a 'nonlocal box', that is, systems obeying the marginal distribution law but violating Bell's inequalities beyond Tsirelson's bound [30]. Following our general perspective in this paper, we call our quantum modeling a 'nonlocal box modeling'. To conclude, we can say that our construction in this section shows that it is possible to elaborate a Hilbert space modeling for nonlocal boxes by introducing suitable entangled measurements, contrary to what is usually believed in quantum foundation circles.

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